Exam Seat No:________________ Enrollment No:____________________

C.U.SHAH UNIVERSITY

Wadhwan City

Subject Code : 5SC02MTC3 **Summer Examination-2014** **Date: 13/06/2014**

Subject Name:- Functional Analysis-I

Branch/Semester:- M.Sc(Mathematics)/II **Time:02:00 To 5:00**

Examination: Regular

Instructions:-

(1) Attempt all Questions of both sections in same answer book / Supplementary

(2) Use of Programmable calculator & any other electronic instrument is prohibited.

(3) Instructions written on main answer Book are strictly to be obeyed.

(4) Draw neat diagrams & figures (If necessary) at right places

(5) Assume suitable & Perfect data if needed

SECTION-I

Q-1 a) Define normed linear space. (01)

b) Define separable space. (01)

c) Let X be an inner product space. For $x \in X$, define $\|x\|: X \to R$, as (02)

$$\|x\| = \sqrt{<x, \; x>}.$$

Then show that $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

d) Define weakly convergent sequence. (01)

e) Let X be a normed linear space and $T: X \to X$ be linear. If T is continuous at 0, then show that T is continuous on X. (02)

Q-2 a) Show that l^2 is an inner product space with the inner product, for $x = (x_j)_{j\geq 1}$, $y = (y_j)_{j\geq 1} \in l^2, <x, \; y> = \sum_{j=1}^{\infty} x_j \bar{y}_j$. (05)

b) State and prove Schwarz's inequality for an inner product space. When does the equality hold? (05)

c) If X be an inner product space and $\{u_1, u_2, u_3, \dots, u_n, \dots\}$ be an orthonormal subset of X. Let $\{\lambda_n\}$ be a sequence in K. If $\sum_{n=1}^{\infty} \lambda_n u_n$ converges to $x$in X, then prove that $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$ and $\lambda_n = \langle x, \; u_n \rangle$. (04)

OR

Q-2 a) State and prove Bessel's inequality. (05)

b) Let $x_n(t) = t^n$, $n = 0, 1, 2, 3, \dots; t \in [-1, \; 1]$, then $\{x_0, x_1, x_2, \dots\}$ is a linearly independent subset of the set of all Riemann integrable functions on $[-1, \; 1]$, i.e. $R_w^2[-1, \; 1]$. Also let $w(t) = 1$.Using Gram-Schmidt orthonormalization process covert this set into orthonormal set, where $\langle f, \; g \rangle = \int_{-1}^{1} f \; \bar{g} \; w \; dt$, for $f, g \in R_w^2[-1, \; 1]$. (05)

c) Prove that $x_n \to x$ in H iff (x_n) convergence weakly to $x$and $\|x_n\| \to \|x\|$. (04)

Q-3 (a) Let H be a Hilbert space. Then show that the following are equivalent. (07)

(i) H has countable orthonormal basis.

(ii) H is isometric isomorphic to K^n or l^2.

(iii) H is separable.

(b) State and prove Riesz representation theorem. Also give an example which shows that completeness is essential for it. (07)

OR

Q-3 a) State and prove projection theorem. (07)

b) If $x_n \to x$ in H iff $\langle x_n, y\rangle \to \langle x, y\rangle$ uniformly for $y \in H, \|y\| \leq 1$. (04)

c) For $f, g \in H'$, define $\langle f, g\rangle' = \langle y_g, y_f\rangle$, where y_g is the representator of f. Then prove that H' is a Hilbert space. (03)

SECTION-II

Q-4 a) Define self-adjoint operator and normal operator. (02)

b) If A and B are unitary operators on H, then show that AB is also an unitary operator. (02)

c) Define approximate eigen value of an operator $A \in BL(H)$. (01)

d) Show that, if A is normal then, $\lambda \in \sigma_e(A)$ iff $\bar{\lambda} \in \sigma_e(A^*)$. (02)

Q-5 a) Let $A \in BL(H)$. Then $\exists$ unique self adjoint operators $B, C \in BL(H)$ such that $A = B + iC$. (05)

b) Let $A \in BL(H)$. Then show that there exists unique $B \in BL(H)$ such that $< Ax, y > = < x, By >$, for all $x, y \in H$. (05)

c) Show that $\sigma_e(A) \subset \sigma_a(A) \subset \sigma(A)$, for $A \in BL(H)$. (04)

OR

Q-5 a) Let $A \in BL(H)$. Prove that A^* is bounded below iff $R(A) = H$. (05)

b) Let $H = l^2$. Define $A: H \to H$, as $A(x(1), x(2), x(3), \dots) = (0, x(1), x(2), x(3), \dots)$. Then find matrix of A with respect to standard basis of H and using matrix find $A^*(x(1), x(2), x(3), \dots)$. (05)

c) Give an operator A for which $\sigma_e(A)$ is empty. (04)

Q-6 a) Let $A \in BL(H)$. Prove that A is unitary iff A is isometry $(\|Ax\| = \|x\|)$ and onto. (05)

b) Let $A: H \to H$ be linear operator. The prove that A is compact iff $\overline{A(U)}$ is a compact subset of H, where $U = \{x \in H: \|x\| < 1\}$. (05)

c) Let A, B be compact operators on H. Then $A + B, \alpha A(\alpha \in K)$ are compact. (04)

OR

Q-6 a) Let H be a separable Hilbert space. Let $A \in BL(H)$ be a Hilbert Schmidf operator, then prove that A is compact. (07)

b) Let H be a non-zero Hilbert space and A be a compact operator on H. Then prove that every non-zero approximate eigen value of A is an eigen value of A and the dimension of the eigen space is finite corresponding to non-zero eigen value. (07)

*********13***14****S**